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Energy Management in Rocket Propulsion

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The optimal management of the available energy in an ideal single-stage rocket produces a velocity increment that is larger than the value provided by Tsiolkovsky's equation. This result is immediately applicable when a limited amount of energy is available from a source that is external to the propellant. A storage device is instead necessary to delay partially the use of the energy that is produced by the combustion of a reactive working fluid. The penalty of the energy-source mass and the presence of an upper limit on the propellant temperature are also discussed; in the latter case, the addition of a low-molecular-weight, inert propellant to the reactive propellant is beneficial during the final phase of the rocket acceleration.

Nomenclature

c	= effective exhaust velocity
E	= used energy
F	= available chemical energy
H	= Hamiltonian function, [see Eqs. (3) and (18)]
m	= mass fraction
V	= rocket velocity
x	= reactive propellant fraction
β	= specific energy
ΔV	= ideal velocity increment
λ_E, λ_F, λ_V	= adjoint variables

Subscripts

f	= final value
i	= initial value
m	= uniform energy supply value
max	= maximum allowable value
p	= propellant
r	= rocket
u	= payload and structure

Superscript

$'$	= inert propellant

Introduction

RECENTLY King[1] analyzed the improvement of the rocket velocity increment that could be achieved by transferring energy from a reactive propellant to an inert, low-molecular-weight gas that is expelled in the last phases of the acceleration process. The goal of this procedure is to obtain better matching of the propellant exit velocity with the changing vehicle velocity; a favorable consequence of leaving the exhaust gases almost motionless with respect to the surroundings would be an improvement of the velocity increment. The problem of minimizing the exhaust kinetic energy losses is usefully considered in aircraft propulsion but is seldom addressed in rocket propulsion. The present analysis has been started to deal with the specific problem of King's work[1] using a less pragmatic and more theoretical approach; it has eventually been directed to the general aspects of energy management to improve rocket performance. The theory of optimal control is the main tool of the analysis and is used to highlight the basic physics of the problem. The optimization of a practical rocket system depends on a greater number of parameters; the search for optimal performance is a complex problem that is not in the scope of the present paper.

Rocket propulsion is rapidly evolving; in particular, new techniques are being introduced to accelerate the propellant and a separate powerplant often supplies the required energy to an inert propellant. This paper searches for the ideal velocity increment of a single-stage rocket when a limited amount of energy is available from a source that is external to the propellant. If a power-limited electric-propulsion system is used, this problem is well posed when the operation time is bounded and when the power limit is actually an energy limit. It is assumed that the energy can be converted into kinetic energy of the propellant without any losses; the effective exhaust velocity is not bounded because the acceleration technique of the working fluid is not specified. Even though the most interesting results concern advanced propulsion, a more conventional chemical-propulsion system is also analyzed to deal with the same problem,[1] which prompted the present analysis. In this case, the rocket inert mass comprises a highly idealized storage device that, independently of its practical feasibility, transfers energy from the burning propellant to the fluid that is subsequently exhausted.

Unconstrained Problem

The ideal acceleration of a rocket in a vacuum, in the absence of gravitational and aerodynamical losses, is considered here. The propellant is accelerated without any kind of loss and leaves the rocket after all of the available energy has been converted into kinetic energy. The available mass of the propellant is assigned; the same analysis is applied to two different propulsion systems. An energy source that is external to the propellant is first considered, and $c_m^2/2$ is the available energy per unit mass of propellant; c_m would be the effective exhaust velocity if the energy were uniformly supplied to the propellant. A nonuniform supply rate is instead used to obtain a variable exhaust velocity. The second case considers a reactive propellant where $c_m^2/2$ is its specific energy; in this case, an energy storage device is necessary to eject the propellant with an exhaust velocity that is different from c_m.

The exhaust velocity is the process control variable that is used to maximize the final velocity of the rocket. For convenience, masses are expressed as fractions of the initial mass of the rocket, and the amount of ejected propellant is assumed to be the independent variable.[2] Two differential equations rule the problem:

$$\frac{\mathrm{d}V}{\mathrm{d}m_p} = \frac{c}{m_r} = \frac{c}{1 - m_p} \tag{1}$$

$$\frac{\mathrm{d}E}{\mathrm{d}m_p} = \frac{c^2}{2} \tag{2}$$

The initial ($m_{pi} = 0$) value V_i of the rocket velocity is assigned; the total mass of propellant m_{pf} is also assigned. Equation (2) provides

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the energy E that has been used to discharge the propellant mass m_p. The initial value E_i is obviously zero; the assigned final value $E_f = m_{pf} c_m^2/2$ guarantees that the available energy has been completely used.

The Hamiltonian

$$H = \lambda_V[c/(1 - m_p)] + \lambda_E(c^2/2) \tag{3}$$

is introduced to apply the theory of optimal control. Equation (3) is not an explicit function of the state variables V and E; the adjoint variables λ_V and λ_E are constant, according to the Euler–Lagrange equations. In particular, $\lambda_V = 1$ is a necessary condition for optimality; nevertheless, a different value could arbitrarily be assumed because the problem is homogeneous in the adjoint variables. The optimal value of the control is obtained by nullifying the first derivative of the Hamiltonian with respect to the control itself. One now easily obtains

$$c = -\frac{1}{\lambda_E(1 - m_p)} \tag{4}$$

If the optimal control law is adopted, Eqs. (1) and (2) can be integrated analytically to provide

$$V = V_i - (1/\lambda_E)(1/m_r - 1) \tag{5}$$

$$E = \left(1/2\lambda_E^2\right)(1/m_r - 1) \tag{6}$$

By comparing the assigned final value of the energy to the value provided by Eq. (6), one can find the adjoint variable

$$\lambda_E = -\frac{1}{c_m \sqrt{m}_{rf}} \tag{7}$$

This is inserted into Eq. (5) to provide the ideal velocity increment

$$\Delta V = c_m\left[(1 - m_{rf})\big/\sqrt{m}_{rf}\right] \tag{8}$$

which is larger than Tsiolkovsky's value $-c_m \ln m_{rf}$ (Fig. 1), because the historical constraint of constant exhaust velocity does not hold in advanced propulsion and has been removed. Equation (8) may be rearranged in the form

$$m_{rf} = 1 - (\Delta V/2c_m)\left[\sqrt{(\Delta V/c_m)^2 + 4} - (\Delta V/c_m)\right] \tag{9}$$

which states the maximum value of the rocket final mass for an assigned velocity increment in the case of chemical propulsion when c_m is a propellant characteristic.

A better performance is achieved as the kinetic energy losses are minimized by ejecting the propellant with a constant velocity relative to the surroundings:

$$c - V = -(1/\lambda_E) - V_i = c_m \sqrt{m}_{rf} - V_i \tag{10}$$

The kinetic energy losses are completely absent only if $c_m \sqrt{(m_{rf})} = V_i$. The velocity increment is independent of V_i and is favored by a larger availability of propellant and energy to accelerate the propellant with respect to the vehicle. A further resource, that is, the initial level of the propellant kinetic energy with respect to the surroundings, is relevant to ensure energy conservation without any influence on the optimal control law and velocity increment. A mutual relationship between the three resources is needed for zero kinetic energy losses; if a larger amount of specific energy $c_m^2/2$ is available, the best rocket performance is obtained by an exceeding exhaust velocity even though a fraction of the available energy is wasted in space. In the case of a propellant excess, it is convenient to discharge the whole mass with a lower velocity, while accepting the corresponding kinetic energy losses.

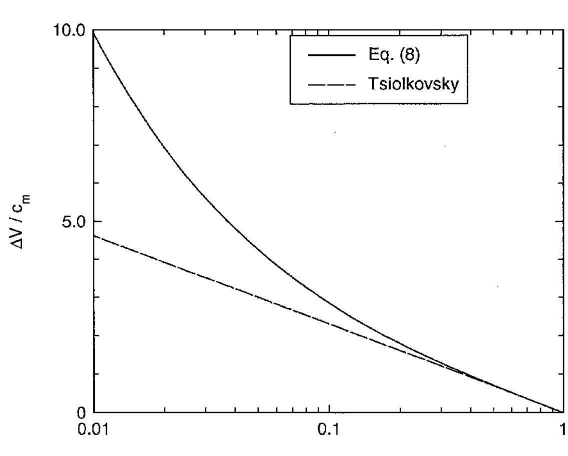

Fig. 1 Ideal velocity increment for a single-stage rocket.

Power Supply Penalty

An electric powerplant is heavy, and the influence of its mass on the rocket performance is usually taken into account.[3] When the system operation time is assigned, the available energy can be considered to be proportional to the powerplant mass:

$$E_f = \beta(1 - m_u - m_{pf}) \tag{11}$$

where the assigned mass m_u comprises the payload and structures. In the problem analyzed in this section, the total propellant energy is to be chosen to maximize the final velocity. Equation (11) relates the independent variable and energy final values; the corresponding necessary condition for optimality $H_f = -\beta\lambda_E$ is furnished by the theory of optimal control. The use of Eq. (3) with the optimal control law given by Eq. (4) provides the constant value

$$\lambda_E = -\frac{1}{\sqrt{2\beta} m_{rf}} \tag{12}$$

Using this value, one can compare Eq. (6) with Eq. (11) and obtain $m_{rf} = \sqrt{(m_u)}$, that is, the optimal share of the available mass between the propellant and powerplant, which is found to be independent of the velocity increment and power-source characteristics, as in the classical analysis.[3] The field of validity for this result is probably larger than the present problem; for instance, the same optimal share has been found by the authors when dealing with a more complex and practical problem of electric-propulsion optimization.[4] Equation (5) gives the rocket velocity increment

$$\Delta V = \sqrt{2\beta}\left(1 - \sqrt{m}_u\right) \tag{13}$$

when the mass of the energy source is not negligible.

Note that Eq. (8) still holds but, according to its definition,

$$c_m = \sqrt{\frac{2E_f}{m_{pf}}} = \sqrt{2\beta}\sqrt{\frac{m_{rf} - m_u}{1 - m_{rf}}} \tag{14}$$

is now a function of the rocket final mass, that is, of the propellant mass. Equation (8) can be rewritten as

$$\Delta V = \sqrt{2\beta}\sqrt{\frac{(m_{rf} - m_u)(1 - m_{rf})}{m_{rf}}} \tag{15}$$

and the results of its application are presented in Fig. 2 (solid line). Equation (13) directly provides the maximum velocity increment corresponding to the optimal value of the propellant mass, which can alternately be obtained by means of an ordinary differentiation of Eq. (15).

Constrained Problem

The thermodynamic acceleration of the heated propellant in a conventional nozzle is assumed in this section. The square of the exhaust velocity is inversely proportional to the molecular weight of the propellant, for an assigned value of the total temperature in the nozzle; this temperature is usually limited for practical reasons. When the energy source is external to the propellant, one should search for the lowest molecular weight. Both the chemical energy and molecular weight must instead be considered in the case of a reactive propellant. Once the propellant has been selected, the limit

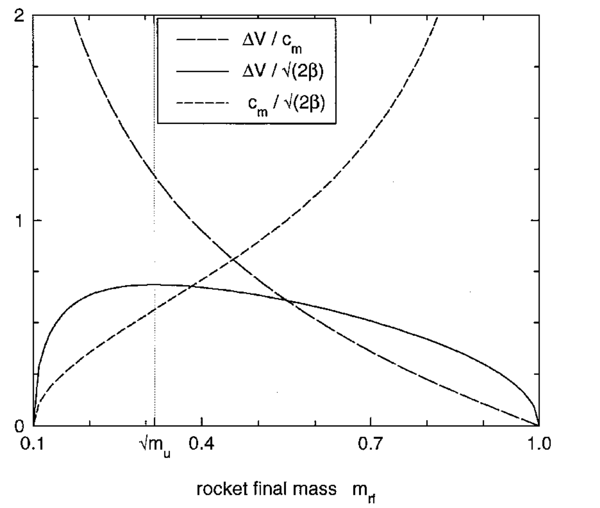

Fig. 2 Rocket performance for different shares of the available mass between the powerplant and propellant (m_u = 0.1).

on the total temperature can easily be stated as a limit value $c_{\max}$ of the exhaust velocity.

If $c_{\max} < c_m$ the entire available energy cannot be exploited, and the maximum acceleration is obtained with $c = c_{\max}$, that is, by using as much energy as possible. On the contrary, it is possible to use all of the energy when $c_{\max} \geq c_m$; the optimal solution is constituted by an initial phase with c provided by Eq. (4), until $c = c_{\max}$ is attained. A maximum-temperature, constant-c phase follows.

In the case where a reactive propellant is used, the addition of a low-molecular-weight, inert propellant, which has been heated by means of the stored energy, could be convenient in the final phase of the rocket acceleration.[1] This technique is particularly interesting when $c_{\max} < c_m$ and the exceeding energy would not be used otherwise. The problem is now ruled by two constrained controls, the mass fraction of the reactive propellant $0 \leq x \leq 1$ and the effective exhaust velocity c that is subject to the maximum temperature limit. A rigorous application of the theory of optimal control[5] to this problem is omitted here for the sake of conciseness. The optimal solution is composed of three phases; only the reactive propellant is used during the initial phases, first with c given by Eq. (4) and then with $c = c_{\max}$. In the third phase, the reactive and inert propellants are heated to the maximum temperature and mixed before their expansion in the nozzle. An enthalpy balance provides the relation

$$x = \frac{c'^2_{\max} - c^2}{c'^2_{\max} - c^2_{\max}} \tag{16}$$

between the reactive propellant fraction and the effective exhaust velocity, which is improved as, at the same temperature, the propellant with the lower molecular weight presents a higher exhaust velocity ($c'_{\max} > c_{\max}$).

A differential equation that expresses the energy obtained from the propellant:

$$\frac{\mathrm{d}F}{\mathrm{d}m_p} = x\frac{c_m^2}{2} \tag{17}$$

is added to Eqs. (1) and (2) with the obvious boundary conditions $F_i = 0$ and $F_f = E_f$; because of the presence of the latter condition, the theory of optimal control provides $\lambda_E = -\lambda_F$ and $\lambda_V = 1$ is again assumed. The Hamiltonian becomes

$$H = c/(1 - m_p) + (\lambda_E/2)\left(c^2 - x c_m^2\right) \tag{18}$$

and the nullity of its derivative during the third phase, when x is no longer a unit but is instead given by Eq. (16), provides the optimal control law

$$c = -\frac{1}{\lambda_E(1 - m_p)}\left(\frac{c'^2_{\max} - c^2_{\max}}{c'^2_{\max} - c^2_{\max} + c_m^2}\right) \tag{19}$$

Table 1 Optimal velocity increment

$c_{\max}/c_m$	$c'_{\max}/c_m$	Monopropellant, $\Delta V/c_m$	Bipropellant, $\Delta V/c_m$	Benefit, %
0.8	3.2	1.842	2.768	50.3
0.9	3.6	2.072	2.787	34.5
1.0	4.0	2.303	2.800	21.6
1.1	4.4	2.414	2.810	16.4
1.2	4.8	2.501	2.817	12.6

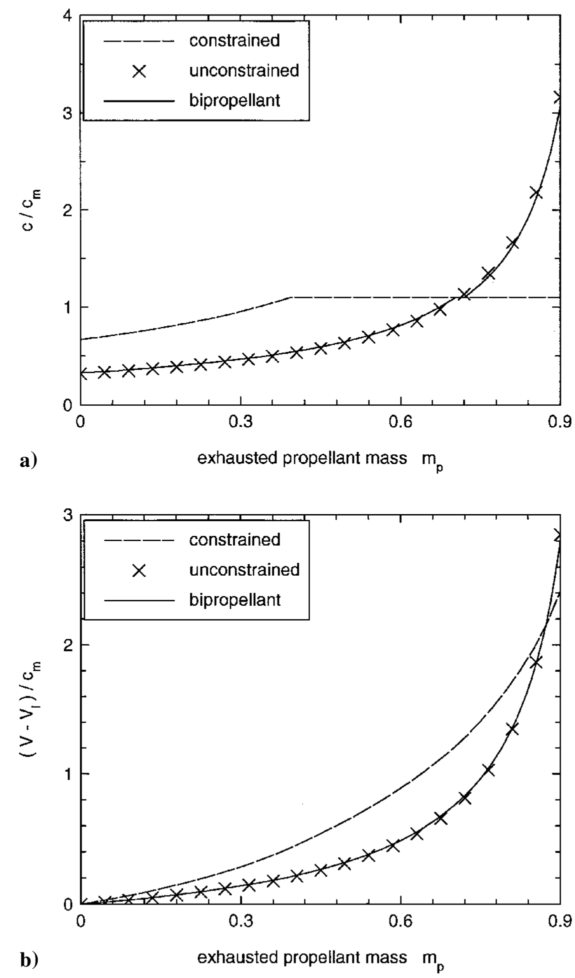

Fig. 3 Comparison of optimal control laws and corresponding rocket accelerations (m_{rf} = 0.1, $c_{\max}$ = 1.1 c_m, and $c'_{\max}$ = 4 $c_{\max}$, when applicable).

The control law and the progressive velocity increment for bipropellant operation (solid lines) are compared in Fig. 3 with the unconstrained (symbols) and constrained (dotted lines) monopropellant operation; $m_{rf} = 0.1$, $c_{\max} = 1.1\, c_m$, and $c'_{\max} = 4\, c_{\max}$ have been assumed. The addition of the low-molecular-weight propellant allows one to follow closely the optimal unconstrained control law, while respecting the temperature limit. Table 1 presents the rocket velocity increment for different values of $c_{\max}$, that is, of the maximum allowable temperature. The greatest benefit of the inert propellant addition is connected to the lowest temperature limit.

Conclusions

Tsiolkovsky's equation provides the velocity increment of an ideal rocket when the available energy is uniformly supplied to the propellant. A larger velocity increment can be achieved if the exhaust velocity is progressively increased during the acceleration process, thus minimizing the kinetic energy losses in space. A simple equation expresses this larger velocity increment of a rocket operating in a vacuum, when the mass of the energy source is negligible.

Another simple expression applies if the powerplant mass is taken into account. The same concepts, when applied to a chemical rocket, confirm that performance is improved by the addition of a low-molecular-weight, inert propellant during the final phase of the acceleration process.

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